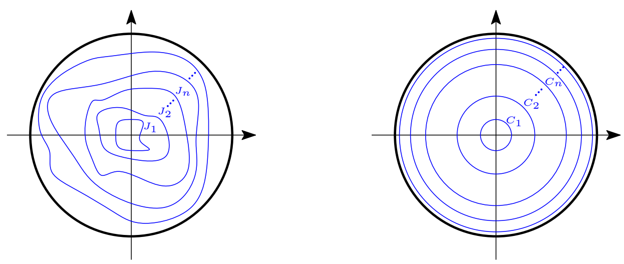
Flip a Coin, Get an Annular Function?

This project is based on an article that was published in the *American Mathematical Monthly* (February, 2025, doi: https://doi.org/10.1080/00029890.2024.2419304). It was authored by three Westmont College Students (Curtis Barnhart, Isaac Jessop, Sam Tang) and their faculty advisor (Russell Howell). Following is the relevant information.

1. Background

A function f analytic in the open unit disk D is said to be *annular* if there is a sequence $\{J_n\}$ of nested Jordan curves converging outward to the unit circle such that $\lim_{n\to\infty}\left(\min_{z\in J_n}|f(z)|\right) = \infty$. If the J_n can be taken to be concentric circles C_n, then f is *strongly annular*.



Now, pick a number at random between 0 and 1 and write it in its binary decimal form.

Examples: $\frac{1}{4} = 0.01000000\ldots$; $\frac{1}{3} = 0.01010101\ldots$

Associate that number with an infinite series of the form $\Sigma \pm z^n$, where the plus or minus signs are determined by the corresponding digits in the binary expansion of the number chosen: $0 \longrightarrow +$; $\quad 1 \longrightarrow -$.

Examples:

$\frac{1}{4} = 0.01000000\ldots$ gives $\quad \Sigma \pm z^n = 1 - z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + \cdots$;

$\frac{1}{3} = 0.01010101\ldots$ gives $\quad \Sigma \pm z^n = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + \cdots$.

2. The Project

What is the probability that the number picked results in an infinite series that is strongly annular? In the *Monthly* article we proved that the probability is either zero or one (independently of the zero-one law), and also found uncountable collections of numbers giving rise to series that have and do not have that property. So, what is the probability—zero or one?

3. Contact Details

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